Solution Sheet 12

Exercise 12.1

Let $f_n, f : \mathcal{X} \to \mathcal{Y}$ be measurable maps on Banach spaces and such that $f_n(x_n) \to f(x)$ for any sequence x_n in $\mathcal{X}$ converging to x. Prove that f is continuous and f_n converges to f locally uniformly.

Proof. Suppose that f has a discontinuity at x, then there exists $\delta > 0$ and $y_n \to x$ such that

$$|f(y_n) - f(x)| \geq \delta.$$

Let $N_0 = 1$, and since $f_n(y_k) \to f(y_k)$

$$N_k = \inf\{N \geq N_{k-1} : |f_n(y_k) - f(x)| \geq \frac{\delta}{2}, \forall n \geq N\} < \infty.$$

Define $x_n = y_k$ if $n \in [N_k, N_{k+1})$ then $x_n \to x$. But $|f_n(x_n) - f(x)| \geq \frac{\delta}{2}$ by the construction, contradicting with the assumption and proving that f must be a continuous functions.

Suppose that f_n does not converge locally uniformly. Then for any relatively compact set K, there exists $\delta > 0$ and $x_n \in K$ such that

$$|f_{n_k}(x_k) - f(x_k)| > \delta.$$

Now x_k has a convergent subsequence, which we denote by y_k, with limit y. We have

$$|f_{n_k}(y_k) - f(y_k)| > \delta.$$

Since f converges locally uniformly, there exists N with $|f(y_n) - f(y)| < \frac{\delta}{2}$ for any $k > N$ and

$$|f_{n_k}(y_k) - f(y)| > \delta/2,$$

contradicts the assumption. We have showed therefore for any K compact , any δ there exists N such that $|f_n(x) - f(x)| < \delta$ for all $x \in K$. □

Exercise 12.2

Prove the following Continuous Mapping Theorem:

Let $f_n, f : \mathcal{X} \to \mathcal{Y}$ be measurable maps between metric spaces such that $f_n(x_n) \to f(x)$ for any sequence x_n in $\mathcal{X}$ converging to x. If $\mu_n, \mu \in \mathbb{P}(\mathcal{X})$ with $\mu_n \to \mu$, Then $(f_n)_*\mu_n \to f_*\mu$. In particular, if ξ_n are random variables converging to ξ in distribution, then $f_n(\xi_n)$ converges to $f(\xi)$ in distribution.

Proof. Denote $\nu_n = (f_n)_*\mu_n$ and $\nu = f_*\mu$. By Portmanteau theorem it is sufficient to show for any $G \subset \mathcal{Y}$ open,

$$\liminf_{n\to\infty} \nu_n(G) \geq \nu(G).$$

Fix such an open set $G \subset \mathcal{Y}$. For any $x \in f^{-1}(G)$, there exists a neighbourhood U and a number m such that for all $k \geq m$, $f_k(U) \subset G$. Consequently, $x \subset \cap_{k=m}^{\infty}(f_k^{-1}(G))^o$, where the superscript denotes the interior of a set, in particular it is an open set. Thus,

$$f^{-1}(G) \subset \cup_{m=1}^{\infty} \cap_{k=m}^{\infty} (f_k^{-1}(G))^o.$$

Consequently,

$$f_*\mu(G) = \mu(f^{-1}(G)) \leq \sup_m \mu(\cap_{k=m}^{\infty}(f_k^{-1}(G))^o) \leq \sup_m \liminf_{n\to\infty} \mu_n(\cap_{k=m}^{\infty}(f_k^{-1}(G))^o)).$$

We have used $\mu_n \to \mu$. Finally we obtain,

$$f_*\mu(G) \leq \sup_m \liminf_{n\to\infty} \mu_n(\cap_{k=m}^{\infty}(f_k^{-1}(G))^o)) \leq \liminf_{n\to\infty} \mu_n(f_n^{-1}(G)) = \liminf_{n\to\infty} (f_n)_*\mu_n(G),$$

which completes the proof. □

Exercise 12.3

Let P be a transition function on $\mathcal{X}$ and let $V\colon \mathcal{X} \to \mathbb{R}_+ \cup \{\infty\}$ be a Borel measurable function. Suppose there exist a positive constant $\gamma \in (0,1)$ and a constant $C > 0$ such that

$$TV(x) \leq \gamma V(x) + C\ ,$$

for every x such that $V(x) \neq \infty$. Then

$$T^n V(x) \leq \gamma^n V(x) + \frac{C}{1-\gamma}.$$

Proof. This is a consequence of the Chapman-Kolmogorov equations:

$$\begin{aligned} T^n V(x) &= \int_{\mathcal{X}} V(y)\, P^n(x,dy) = \int_{\mathcal{X}} TV(y) P^{n-1}(x,dy) = \int_{\mathcal{X}}\int_{\mathcal{X}} V(y)\, P(z,dy)\, P^{n-1}(x,dz) \\ &\leq C + \gamma \int_{\mathcal{X}} V(z)\, P^{n-1}(x,dz) \leq \ldots \\ &\leq C + C\gamma + \ldots + C\gamma^n + \gamma^n V(x) \leq \gamma^n V(x) + \frac{C}{1-\gamma}\ , \end{aligned}$$

completing the proof. □

Exercise 12.4

Let $F : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ be continuous and bounded, (ξ_n) a collection of i.i.d $\mathcal{Y}$-valued random variables independent of the $\mathcal{X}$-valued random variable X_0. Define

$$X_{n+1} = F(X_n, \xi_n).$$

Suppose, furthermore, there exists a Borel measurable function $V\colon \mathcal{X} \to \mathbb{R}_+$ with compact sub-level sets and constants $\gamma \in (0,1)$ and $C \geq 0$ such that

$$\int_{\mathcal{Y}} V\big(F(x,y)\big)\, \hat{\mathbb{P}}(dy) \leq \gamma V(x) + C\ , \quad \forall x \in \mathcal{X}\ ,$$

where $\hat{\mathbb{P}}$ is the distribution of ξ_n. Prove that the process X has at least one invariant probability measure.

Proof. We recall from Exercise 7.1 that X was shown to induce a Feller semigroup. Moreover,

$$P(x,A) = \mathbb{E}(X_1 \in A | x_0 = x) = \mathbb{E}(F(X_0,\xi_0) \in A | X_0 = x) = \int_{\mathcal{Y}} \mathbb{1}_A(F(x,y)) \hat{\mathbb{P}}(dy).$$

Then the left hand side of the given inequality is TV and V is a Lyapunov function. The existence of an invariant probability measure now follows from the Lyapunov function test. □